Start

$$Ax = b \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

1. Column ($j = 1$): Assume the *pivot element* $a_{11} \neq 0$. We define

$$\ell_{i1} := \frac{a_{i1}}{a_{11}} \quad , \quad i = 2, \dots, m$$

and build the matrix called GAUSS elimination matrix

$$G_1 := L_{m1}(-\ell_{m1}) \cdots L_{31}(-\ell_{31}) L_{21}(-\ell_{21})$$

or written

$$G_1 = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -\ell_{21} & 1 & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\ell_{m1} & 0 & \dots & 0 & 1 \end{bmatrix} = I - \sum_{i=2}^{m} \ell_{i1} e_i e_1^T$$

We get $G_1 A x = G_1 b \Leftrightarrow: A^{(1)} x = b^{(1)} \Leftrightarrow$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & \dots & a_{mn}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix}$$

2. Column ($j = 2$): Assume that the pivot element $a_{22}^{(1)} \neq 0$. We define

$$\ell_{i2} := \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} \quad , \quad i = 3, \dots, m$$

and build

$$G_2 := L_{m2}(-\ell_{m2}) \cdots L_{42}(-\ell_{42}) L_{32}(-\ell_{32})$$

or in matrix form

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & & & \vdots \\ 0 & -\ell_{32} & 1 & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & -\ell_{m2} & 0 & \dots & 0 & 1 \end{bmatrix} = I - \sum_{i=3}^{m} \ell_{i2} e_i e_2^T$$

We get $G_2 A^{(1)} x = G_2 b^{(1)} \Leftrightarrow: A^{(2)} x = b^{(2)}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{123} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & 0 & a_{33}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{m3}^{(2)} & \dots & a_{mn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(2)} \\ b_2^{(2)} \\ \vdots \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

And so on until $j = n-1$.

At the end of this procedure we have a upper triangular matrix

$$R := A^{(n-1)} = G_{n-1} \cdots G_1 A$$

where

$$G = I - \sum_{j=1}^{n-1} \sum_{i=j+1}^{m} \ell_{ij} e_i e_j^T$$

and the right side

$$c := b^{(n-1)} = G_{n-1} \cdots G_1 b \quad .$$

Using backward elimination to get the solution of the LES

$$Rx = c \quad .$$

At the end of the Gauss elimination process we get the right triangular matrix

$$R := A^{(n-1)} = G_{n-1} \cdots G_1 A$$

where the product

$$G_{n-1} \cdots G_1 = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -\ell_{21} & 1 & \ddots & & \vdots \\ -\ell_{31} & -\ell_{32} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ -\ell_{m1} & -\ell_{m2} & \dots & -\ell_{m,n-1} & 1 \end{bmatrix} = I - \sum_{j=1}^{n} \sum_{i=j+1}^{m} \ell_{ij} e_i e_j^T$$

is a nonsingular unit left triangular matrix with the inverse matrix

Bsp:

$$Ax = b \Leftrightarrow \begin{bmatrix} 2 & 4 & -4 & 1 \\ -1 & 1 & 2 & 3 \\ 3 & 6 & 1 & -2 \\ 1 & 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -7 \\ 2 \end{bmatrix}$$

```
G1 =

    1.0000         0         0         0
    0.5000    1.0000         0         0
   -1.5000         0    1.0000         0
   -0.5000         0         0    1.0000

>> G2=[1 0 0 0;0 1 0 0;0 0 1 0;0 1/3 0 1]

G2 =

    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0
         0    0.3333         0    1.0000

>> G3=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 2/7 1]

G3 =

    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0
         0         0    0.2857    1.0000

>> G=G3*G2*G1

G =

    1.0000         0         0         0
    0.5000    1.0000         0         0
   -1.5000         0    1.0000         0
   -0.7619    0.3333    0.2857    1.0000
```

Allerdings steht unten links in G nicht $-l_{41} = -0.5$